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ARTICLES

ENDOGENOUS GROWTH, BACKSTOP TECHNOLOGY ADOPTION, AND OPTIMAL JUMPS

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This paper analyzes a two-phase endogenous growth model in which the adoption of a backstop technology (e.g., solar) yields a sustained supply of essential energy inputs previously obtained from exhaustible resources (e.g., oil). Growth is knowledge-driven and the optimal timing of technology switching is determined by welfare maximization. The optimal path exhibits discrete jumps in endogenous variables: technology switching implies sudden reductions in consumption and output, an increase in the growth rate, and instantaneous adjustments in saving rates. Due to the positive growth effect, it is optimal to implement the new technology when its current consumption benefits are substantially lower than those generated by old technologies.

Keywords: Backstop Technology, Discrete Jumps, Endogenous Growth, Exhaustible Resources, Optimal Control

1. INTRODUCTION

One of the major challenges facing modern economies is the sustainability problem induced by resource dependence. Despite the rapid development guaranteed by technical progress, the production process of postindustrial economies still relies on a finite supply of minerals and fossil fuels, and the question of how to preserve individual welfare in the future is a worldwide political concern. In the past decade, a substantial body of economic literature has tackled the issue of sustainability from the perspective of modern growth theory. Several authors analyzed the conditions under which technological progress is able to guarantee a sustained flow of output when exhaustible resources—e.g., oil—are essential inputs in production. Following the main insights of Stiglitz (1974), these contributions reformulated the problem in the context of endogenous growth models, where the conditions for achieving positive growth rates in the long run are intimately linked to the development of innovations and the profitability of R&D investment [Barbier (1999); Scholz and Ziemes (1999); Bretschger and Smulders (2006)]. In this framework, the allocation mechanism is derived from intertemporal utility maximization *à*

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la Ramsey, and a crucial sustainability condition is that the rate of resource-augmenting technical progress be sufficiently high relative to the utility discount rate [Di Maria and Valente (2008)].[1]

This strand of literature addresses the issue of resource substitution only to some extent. Endogenous growth models exhibit long-run equilibria where production possibilities are sustained by the accumulation of knowledge-type capital. This form of technical progress progressively substitutes the resource in the sense that the increased efficiency of knowledge capital compensates, in terms of productivity, for the restrictions imposed by resource scarcity on production possibilities. Because this mechanism does not make the resource "superfluous" in a finite time, the transitional dynamics of consumption and output are smooth. It may be argued that the substitution process is quite different when perfect substitutes for the resource exist: if the availability of new inputs makes the resource-based technology obsolete, the traditional method of production is abandoned in finite time, and the transition to resource-free techniques may involve nonsmooth dynamics. However, this issue has not been addressed in endogenous growth models, and this is the main motivation for this paper.

At the theoretical level, the analysis of technology adoption in models with exhaustible resources is generally confined to partial equilibrium settings. In this framework, resource scarcity sets limits to economic activity in the long run, and the production process can be perpetuated only by implementing a *backstop technology*—i.e., a new method of production whereby exhaustible natural inputs are replaced by alternative, nonscarce factors [Nordhaus et al. (1973)]. This literature treats the flow of energy as a normal consumption good entering the intertemporal utility function. Agents compare the time profiles of the costs of alternative sources of benefits—say, oil versus solar energy—and decide how much and for how long the exhaustible resource should be consumed. The main predictions of the early models, pioneered by Hoel (1978), Dasgupta and Stiglitz (1981), and Dasgupta et al. (1982), are two: (i) the criterion for backstop technology adoption is *price-based*, because only one technology, the less expensive one, is adopted at any point time; and (ii) the price time-path is *J-shaped*, with increasing energy prices in the short run, and a constant time-profile in the long run. The first result hinges on the assumptions that energy demand and the supply of the backstop technology are stable over time. The energy price equals the marginal cost of the technology in use which, under competitive conditions, is the less expensive on the market. The second result is due to the fact that resource-based energy is exploited in the short run, but increased resource scarcity makes energy prices grow over time. When resource-based energy becomes as expensive as solar-based energy, the latter method is adopted, and the nonscarce nature of solar inputs stabilizes energy prices from that point onward.

Subsequent contributions extended the basic model in various directions. Prediction (i) is modified if the timing of adoption internalizes, due for example to public intervention, the side effects of technology switching. If the use of exhaustible resources generates pollution that reduces private welfare, it may be optimal to switch to the new technology even if its current user cost exceeds that of the old technology, the wedge being determined by the marginal welfare benefit of reduced pollution in the transition [Tahvonen (1997)]. Similarly, it may be argued that the switching time is not determined by the equality between the instantaneous marginal costs of alternative technologies if the backstop is expected to involve substantial technical change. If the cost of the new technology decreases with cumulative use because of learning by doing, a forward-looking agent may decide to adopt the new technology even while the old one is still cheaper. This point is emphasized by Braumollé and Olson (2005) in a model of transition from dirty to clean technologies without exhaustible resources, but results should not change if we assume that the need of backstop technologies stems from resource exhaustibility. This conjecture appears confirmed by the analysis of Tsur and Zemel (2003), who assume that the productivity of new technologies is determined by R&D investment made before adoption. Tsur and Zemel (2003) show that the optimal transition to backstop substitutes is characterized by a "most rapid approach" whereby R&D effort should attain a target level as soon as possible, and the timing of adoption is determined by the effective marginal cost of the backstop—which differs from the instantaneous monetary cost and depends on a knowledge stock that is endogenously accumulated. In a similar vein, a recent paper by Chakravorty et al. (2009) shows that the interplay between learning by doing—which modifies the time profile of the effective marginal costs of the clean backstop—and environmental regulation—which imposes an upper bound on emissions and therefore on fossil-fuel use—generates a mixed composition of energy sources across phases, and thereby a cyclical behavior of energy prices over time.

The above discussion clarifies that, although backstop technology adoption is a crucial issue for analyzing the sustainability problem, resource-based growth and substitute technologies are usually studied within different frameworks. On the one hand, the common denominator of early and recent models of backstop technology adoption, from Hoel (1978) to Chakravorty et al. (2009), is the partial equilibrium setting. On the other hand, endogenous growth models with exhaustible resources typically exclude technology switching. Merging the two approaches is desirable and is likely to generate new results. In endogenous growth models, natural resources are inputs combined with other manmade production factors. Because economic development is driven by investment rates, the resource demand schedule is not stable over time, and the time profile of resource use is crucially affected by the accumulation of the productive stocks representing the engine of growth. In this context, the adoption of a backstop technology would generate a combination of growth effects and level effects in the aggregate: as the economy switches to the new technology, the growth rate is modified because investment productivity depends on whether exhaustible resources are used in production or not. These aggregate effects matter for the optimal timing of technology adoption, but are generally neglected in partial equilibrium models. Building on this point, this paper studies how endogenous growth models currently used in the sustainability literature can provide a more complete criterion for optimizing the timing of backstop technology adoption.

The model presented postulates that aggregate production requires energy, initially produced by means of exhaustible resources like oil. The backstop technology is represented by solar-based energy, and the engine of growth is knowledge accumulation endogenously determined by R&D investment. A benevolent social planner decides whether and when to abandon traditional oil-based energy in favor of the new technology. The model exhibits closed-form solutions for the optimal switching time and the time paths of all endogenous variables. Interestingly, the transition to solar-based energy involves discrete jumps in consumption and output: the adoption of the backstop technology implies a sudden reduction in consumption and output levels, an increase in the growth rate, and instantaneous adjustments in the saving propensity. The intuition for these results is as follows. On the one hand, technology switching implies a transition from slow to fast growth because the resource-based economy is constrained by natural scarcity whereas the solar-based economy fully exploits the growth potential of R&D investment. On the other hand, because the productivity of knowledge accumulation changes between the two phases, there is an intertemporal reallocation effect whereby consumption levels are reduced by technology switching. During the first phase, the traditional technology yields slow growth and the exhaustible resource is exploited to obtain high consumption levels in the short run. When the economy switches to solar-based energy, consumption levels are lower but this is optimal since the negative level effect of technology switching is compensated—in terms of present-value welfare—by the higher growth rate that the economy enjoys from the instant of adoption onward. Hence, due to the positive growth effects of technology switching, the adoption of new solar-based techniques is optimal even though the associated current benefits in terms of consumption are substantially lower.

The plan of the paper is as follows. Section 2 describes the main assumptions of the model and specifies the two-phase optimal control problem. Section 3 analyzes the optimality conditions and derives an explicit expression for the optimal switching time. Section 4 derives the main results regarding the economic consequences of backstop technology adoption. Section 5 discusses the implications of relaxing some of assumptions made in the analysis. Section 6 concludes.

2. ENDOGENOUS GROWTH WITH BACKSTOP TECHNOLOGY

2.1. Assumptions

The general scheme is as follows. Time is continuous and indexed by $t \in [0, \infty)$. Before instant $t = 0$, the economy is resource-based: aggregate production is obtained by means of labor and energy inputs that consist of nonrenewable resources—e.g., oil—extracted from a finite stock. At time $t = 0$ a new technology is available: energy can be obtained by means of a different method of production whereby exhaustible resources are replaced by a nonscarce input—e.g., solar energy. A benevolent social planner, endowed with perfect foresight and full control over allocations, decides whether and when to adopt the solar-based technology. The transition from resource-based to solar-based technologies is irreversible, and may take place at any instant from time zero onward. We denote by $T \in [0, \infty)$ the instant in which this structural change takes place. The possibility of delaying the adoption of the backstop technology after time $t = 0$ implies that the economy will generally experience two different phases over the interval $t \in [0, \infty)$. During phase 1, delimited by $t \in [0, T)$, the economy is still resource-dependent. During phase 2, delimited by $t \in (T, \infty)$, the economy is solar-based.

The reference framework for modeling economic dynamics is provided by endogenous growth theories. In order to keep the analysis tractable, we will use a fairly simple model of balanced growth. Aggregate output is represented by $Y_i = AF_i(E_i, N)$, where A is the state of technology determined by the current stock of knowledge, E is energy, N is labor, and $i = 1, 2$ is the phase index. Output can be either consumed or invested in knowledge-improving activities—e.g., R&D activity—that enhance future production possibilities. The investment rate determines the growth rate of knowledge, $\dot{A}/A$, which is essentially the Hicks-neutral rate of technological progress in the economy. Assuming decreasing marginal returns to energy and labor, this general scheme can be rationalized in terms of several models where the role of the knowledge stock is played by different engines of growth—e.g., human capital accumulation [Lucas (1988)], learning by doing [Romer (1989)], or expanding varieties of intermediate inputs [Barro and Sala-i-Martin (2004)]. In the present context, we assume that the same kind of knowledge $A(t)$ is exploited in both phases, though it may display different productivity levels. The technologies read

$$Y_1(t) = A(t) \cdot F_1(nR(t), N) = A(t) \cdot (nR(t))^{\delta} N^{1-\delta}, \tag{1}$$

$$Y_2(t) = \alpha A(t) \cdot F_2(mG, N) = \alpha A(t) \cdot (mG)^{\gamma} N^{1-\gamma}, \tag{2}$$

where labor N is fixed and inelastically supplied, $R(t)$ is the amount of resources extracted at time t from a finite resource stock, G is the constant flow of solar energy units available in each instant, and n and m are constant coefficients yielding energy-equivalent measures for the flows of resource and solar units, respectively. Parameter $\alpha > 0$ determines whether knowledge A is more productive ($\alpha > 1$) or less productive ($\alpha < 1$) in the second phase relative to the first phase. The productivity parameters δ and γ lie between zero and unity, and are generally different, as the production elasticity of exhaustible resources is not necessarily equal to the production elasticity of solar-based energy.

Technology (1) is exploited in the interval $t \in [0, T)$, whereas technology (2) is used from time T to infinity. In both phases, the aggregate constraint of the economy reads $Y_i(t) = C_i(t) + D_i(t)$, where C_i is consumption and D_i is investment in knowledge-improving activities. This constraint can be imposed by means of the relation

$$c_i\left(t\right) = 1 - d_i\left(t\right), \qquad i = 1, 2, \tag{3}$$

where the propensity to consume, $c_i \equiv C_i/Y_i$, equals one minus the investment rate $d_i \equiv D_i/Y_i$. The engine of growth in each phase is knowledge accumulation. In general, production possibilities are enhanced by virtue of accumulation laws of the type

$$\dot{A}\left(t\right) = \varphi\left(A\left(t\right), d_i\left(t\right)\right), \qquad \partial\varphi/\partial A > 0, \qquad \partial\varphi/\partial d_i > 0,$$

where $\partial\varphi/\partial A > 0$ represents a knowledge-stock effect that is conceptually equivalent to assuming, e.g., increasing returns to human capital accumulation [Lucas (1988)], spillovers from past R&D activity [Aghion and Howitt (1998)], or, more generally, knowledge spillovers [Acemoglu (2002)]. The assumption $\partial\varphi/\partial d_i > 0$ implies that the accumulation of knowledge increases with the economy's rate of investment, consistent with standard models of balanced growth with endogenous R&D expenditures [Grossman and Helpman (1991); Aghion and Howitt (1998)]. In the present context, we will implement the linear specification

$$\dot{A}\left(t\right) = \psi A\left(t\right) d_i\left(t\right), \tag{4}$$

where $\psi > 0$ is a constant proportionality factor. The linear form (4) can be micro-founded in several ways and has the desirable property of eliminating scale effects, consistent with empirical evidence [Barro and Sala-i-Martin (2004)].[2] In the present context, the linear form is particularly useful because it allows us to obtain optimal balanced-growth paths involving no transitional dynamics (see Section 5.1 on this point). Expression (4) allows for differences in the growth rates of knowledge between the two phases, as the use of different technologies generally implies different saving propensities, $d_1 \neq d_2$.

In the first phase, the resource-based technology (1) operates under the constraint imposed by the scarcity of the exhaustible resource. Denoting by S the resource stock, we have

$$-\dot{S}\left(t\right) = R\left(t\right), \tag{5}$$

which says that the instantaneous reduction in the "natural capital" of the economy equals the rate of resource use. The initial stock $S(0) = S_0 > 0$ is taken as given.

2.2. The Social Problem

The analysis focuses on optimal paths determined by the solution of a centralized social problem. The objective is to maximize the present discounted value of the stream of consumption benefits,

$$V = \int_0^\infty U\left(C\left(t\right)\right) e^{-\rho t} dt, \tag{6}$$

where $\rho > 0$ is the social discount rate, and the instantaneous utility takes the isoelastic form

$$U\left(C\left(t\right)\right)=\frac{C\left(t\right)^{1-\sigma}-1}{1-\sigma},\qquad \sigma>0. \tag{7}$$

Our model falls into the class of two-stage optimal control problems with endogenous switching time studied, e.g., in Tomiyama (1985) and Makris (2001). In this framework, the solution is found by implementing the following procedure. Splitting the objective function (6), present-value welfare equals the sum of the substreams of utilities obtained in the two phases, $V = V_1 + V_2$, where

$$V_1=\int_0^T U\left(C_1\left(t\right)\right)e^{-\rho t}dt, \tag{8}$$

$$V_2=\int_T^\infty U\left(C_2\left(t\right)\right)e^{-\rho t}dt. \tag{9}$$

The first step consists of optimizing phase 2 by finding the paths of consumption and knowledge $\{C_2(t), A(t)\}_T^\infty$ that maximize V_2, taking the switching time T as given. In the second step, we derive the set of conditions that are necessary for optimality during phase 1, for a given terminal time T. In the third step, we complete the set of optimality conditions by including the determination of the instant $T = T^*$ in which it is optimal to switch from the resource-based to the solar-based technology. This allows us to obtain the paths of consumption, knowledge, and resource use $\{C_1(t), A(t), R(t)\}_0^{T^*}$ and $\{C_2(t), A(t)\}_{T^*}^\infty$ that maximize social welfare (6). For the sake of exposition, the discussion in the next section will be mainly technical. A more intuitive discussion about the economic consequences of backstop technology adoption will be provided in Section 4.

3. THE OPTIMAL CONTROL PROBLEM

3.1. The Solar-Based Economy

We begin by solving the social subproblem of phase 2, i.e., after the backstop technology has been adopted. This problem consists of maximizing V_2 subject to the aggregate constraint (3), the solar-based technology (2), and the knowledge accumulation rule (4) in each $t \in (T, \infty)$, taking the available knowledge stock $A(T)$ as given, and holding T fixed. Obviously, this part of the solution is relevant only if the solar-based technology is actually adopted—that is, only if T is finite. Given this precondition, we have a standard infinite-horizon problem associated with the present-value Hamiltonian

$$H_2\left(t\right)=U\left(C_2\left(t\right)\right)e^{-\rho t}+\mu_2\left(t\right)\psi A\left(t\right)d_2\left(t\right), \tag{10}$$

where μ_2 is the dynamic multiplier associated with the accumulation law (4). As shown in the Appendix, the necessary conditions for optimality are

$$\mu_2\psi=U'\left(C_2\right)\cdot e^{-\rho t}\left(Y_2/A\right), \tag{11}$$

$$-\dot{\mu}_2 = (1 - d_2) \cdot U'(C_2) \cdot e^{-\rho t} (Y_2/A) + \mu_2 \psi d_2, \quad \textbf{(12)}$$

$$0 = \lim_{t\to\infty} \mu_2(t) A(t). \quad \textbf{(13)}$$

The multiplier μ_2 represents the shadow value of knowledge accumulation, whereas ψ is the implicit rate of return to investment in knowledge. Hence, condition (11) states that the social value of the return to knowledge accumulation, $\mu_2\psi$, must equal the present-value marginal product of knowledge, $e^{-\rho t}(Y_2/A)$, expressed in utility terms. From (11), the co-state equation (12) can be written as $\dot{\mu}_2/\mu_2 = -\psi$, which is the standard Euler condition asserting that the present-value multiplier declines at a rate equal to the rate of return to investment. These optimality conditions yield[3]

LEMMA 1 (Solar-based economy, T given). *In phase 2, the optimal propensity to consume equals*

$$c_2^* = 1 - (\psi - \rho) / (\psi\sigma) \quad \textbf{(14)}$$

in each $t \in (T, \infty)$. Output, consumption and knowledge grow at the constant rate

$$\dot{Y}_2(t)/Y_2(t) = \dot{C}_2(t)/C_2(t) = \dot{A}(t)/A(t) = \frac{1}{\sigma}(\psi - \rho) \quad \textbf{(15)}$$

from $t = T$ onward. The optimal path is well defined if and only if the parameters satisfy

$$\psi(1 - \sigma) < \rho < \psi. \quad \textbf{(16)}$$

Lemma 1 establishes that the solar-based economy exhibits a constant growth rate. It can be shown that the absence of transitional dynamics is due to the linear accumulation function (4) previously assumed. Expression (15) shows that consumption and output evolve according to the standard Keynes–Ramsey rule, where ψ is the implicit interest rate of the economy. Restriction (16) is necessary and sufficient to have $c_2^* \in (0, 1)$, and also guarantees a strictly positive growth rate ($\psi > \rho$). Notice that the absence of transitional dynamics allows us to obtain closed-form solutions for all the endogenous variables during the second phase. In particular, consumption levels are given by[4]

$$C_2(t) = c_2^* \alpha A(T) (mG)^\gamma N^{1-\gamma} e^{(1/\sigma)(\psi-\rho)(t-T)}. \quad \textbf{(17)}$$

Equation (17) shows that consumption levels in the solar-based economy depend on the knowledge stock available at the beginning of the second phase. The optimal level of $A(T)$ is determined by the optimality conditions that characterize the behavior of the resource-based economy, as shown below.

3.2. The Resource-Based Economy

The optimization problem in phase 1 consists of maximizing (8) subject to the aggregate constraint (3), the accumulation rule (4), and the natural resource constraint (5). Because resource extraction must be optimized, the path of the rate of resource use $R(t)$ represents an additional control variable for the social planner. The initial stocks, $A(0) = A_0$ and $S(0) = S_0$, are exogenously given and strictly positive. The present-value Hamiltonian is

$$H_1(t) = U(C_1(t))\, e^{-\rho t} + \mu_1(t)\, \psi A(t)\, d_1(t) - \lambda(t)\, R(t), \tag{18}$$

where μ_1 and λ are the dynamic multipliers associated with (4) and (5), respectively. In the present problem, the terminal state to be imposed on the knowledge stock differs from the usual transversality condition $\mu_1(T)A(T) = 0$. The reason is that knowledge can be transferred to the solar-based economy and exploited during phase 2: if the solar-based technology is adopted in finite time, there is an implicit "bequest" between the two phases and the amount of knowledge left by the resource-based economy at the terminal date T is optimally chosen only if the effects of $A(T)$ on *second-phase* welfare are taken into account. This reasoning has precise formalizations in optimal control theory [Tomyiama (1985); Makris (2001)]:

LEMMA 2. *In the subproblem of phase 1, the terminal conditions*

$$\lim_{t\to\infty} \mu_1(t)\, A(t) = 0 \qquad \text{if } T = \infty, \tag{19}$$

$$\lim_{t\to T^-} \mu_1(t) = \lim_{t\to T^+} \mu_2(t) \qquad \text{if } T < \infty \tag{20}$$

are necessary for optimality.

Lemma 2 can be interpreted as follows. If there is no technology switching, $T = \infty$, the subproblem collapses to a standard infinite-horizon problem where the optimal path of knowledge accumulation is characterized by the transversality condition (19). If the solar-based technology is adopted, the amount of knowledge that the resource-based economy leaves for future use is optimally chosen only when (20) is satisfied. The intuition is that the dynamic multipliers μ_1 and μ_2 represent, in each phase, the marginal social value of an extra unit of knowledge: condition (20) states that the optimal level of knowledge at the switching instant, $A(T)$, must be such that the marginal cost of accumulation in phase 1 equals the marginal benefit from knowledge exploitation in phase 2.

Before applying Lemma 2, we can characterize the solution to the first-phase optimization problem as follows. Assuming that the optimal path of the propensity to consume is interior, $c_1(t) \in (0, 1)$ in each $t \in [0, T)$, the optimal path is characterized by

LEMMA 3 (Phase 1, optimality conditions for given T). *In the resource-based economy, the conditions*

$$\mu_1(t)\, \psi = U'(C_1(t)) \cdot e^{-\rho t}\, (Y_1(t)/A(t)), \tag{21}$$

$$\lambda(t) = U'(C_1(t)) \cdot e^{-\rho t}(1 - d_1(t))\delta(Y_1(t)/R(t)), \tag{22}$$

$$\dot{\mu}_1(t) = -\psi \mu_1(t), \tag{23}$$

$$\dot{\lambda}(t) = 0, \tag{24}$$

$$S_0 = \int_0^T R(t)\, dt \tag{25}$$

are necessary for optimality, where (21)–(24) are valid in each $t \in [0, T)$.

Equations (21) and (23) are similar to (11) and (12) and have the same interpretation. Condition (22) asserts that, in each instant, the marginal social value of extracting the resource, $\lambda(t)$, must equal the present value of the consumed fraction $(1 - d_1(t))$ of the marginal product of resource use, $\delta(Y_1(t)/R(t))$, evaluated in utility terms.[5] Condition (24) shows that the present-value multiplier associated with the resource stock is constant over time: this is the standard Euler condition in cake-eating problems and is indeed associated with a nonrenewable resource stock. Equation (25) follows from the transversality condition on $S(t)$ and establishes that the initial resource stock must equal the sum of resource-use flows extracted during the first phase. In other words, the whole resource stock must be exhausted by the end of phase 1, because leaving unexploited resources in the ground would be suboptimal.

The general implication of Lemma 3 is that consumption and resource use exhibit constant growth rates during phase 1. As shown in the Appendix, the resource-based economy displays

$$\frac{\dot{C}_1(t)}{C_1(t)} = \frac{\psi - \rho(1+\delta)}{\sigma(1+\delta) - \delta}, \tag{26}$$

$$\frac{\dot{R}(t)}{R(t)} = -\frac{\rho - (1-\sigma)\psi}{\sigma(1+\delta) - \delta} = -\phi, \tag{27}$$

in each $t \in [0, T)$, where we have defined the constant

$$\phi \equiv \frac{\rho - (1-\sigma)\psi}{\sigma(1+\delta) - \delta} > 0 \tag{28}$$

in order to represent the speed of resource depletion $-\dot{R}/R$ in a more compact way.[6] Results (26) and (27) are independent of the choice of optimal switching time. With respect to consumption dynamics, it may be noted that expression (26) differs from the Keynes–Ramsey rule (15) holding in phase 2: the two expressions coincide only if $\delta = 0$. The intuition is that, during phase 1, the economy is constrained by the nonrenewable resource stock. As the exhaustible resource is exploited, increased scarcity is compensated for by accumulating knowledge. This implies that resource productivity matters for consumption-saving decisions, and the elasticity parameter δ affects the growth rate of consumption during phase 1. With respect to resource use dynamics, it is possible to derive a closed-form solution for the optimal extraction plan: integrating (27) over the interval

$t \in [0, T)$, and substituting (25), we have

$$R(0) = \frac{S_0 \phi}{1 - e^{-\phi T}} \quad \text{and} \quad R(t) = R(0)\, e^{-\phi t}. \tag{29}$$

The first expression in (29) shows that the initial rate of resource use $R(0)$ increases with the size of the initial stock S_0, and decreases with the length of the first phase T.

Although (26) and (27) provide the basis for analyzing the dynamics of the resource-based economy, the optimal paths of consumption and knowledge during phase 1 are not determined until we impose the terminal conditions stated in Lemma 2. In this regard, we have to distinguish between the limiting case $T = \infty$, and the finite–switching time case $0 < T < \infty$.

Case $T = \infty$. If the solar-based technology is never adopted, we have $T = \infty$, and the accumulation of knowledge is subject to the transversality condition (19). In this case, the economy is permanently resource-based, and is characterized by the following dynamics:

LEMMA 4 (Phase 1, optimal path without switching). *If $T = \infty$, the optimal propensity to consume equals*

$$c_1^*(t) = \frac{1}{\psi} \cdot \frac{\rho - \psi(1 - \sigma)}{\sigma(1 + \delta) - \delta} \tag{30}$$

in each $t \in [0, \infty)$. From (27), we have $\dot{R}(t)/R(t) = -\psi c_1^* < 0$. Output, consumption and knowledge grow at the constant rates

$$\dot{Y}_1(t)/Y_1(t) = \dot{C}_1(t)/C_1(t) = \frac{\psi - \rho(1 + \delta)}{\sigma(1 + \delta) - \delta}, \tag{31}$$

$$\dot{A}(t)/A(t) = \psi\left(1 - c_1^*\right), \tag{32}$$

in each $t \in [0, \infty)$. This path is well defined if and only if parameters satisfy

$$\sigma(1 + \delta) > \delta, \tag{33}$$

$$\psi(1 - \sigma) < \rho < \psi[1 - \delta(1 - \sigma)]. \tag{34}$$

Lemma 4 shows that, if the solar-based technology is never implemented, the resource-based economy exhibits a constant growth rate in each instant. In particular, the consumption propensity is constant over time, and is affected by the degree of resource dependence.

Case $0 < T < \infty$. If the solar-based technology is adopted in finite time, instead, we have $0 < T < \infty$, and the optimal path of knowledge is subject to the terminal condition (20). In this case, the economy must satisfy (see the Appendix)

$$U'(C_1(T))\, Y_1(T) = U'(C_2(T))\, Y_2(T), \tag{35}$$

and the optimal paths of output and knowledge in phase 1 have to be determined simultaneously with the optimal switching time $T = T^*$, given that T^* is finite. It will later be shown that the characteristics of phase 1 with a finite switching time T are identical to those stated in Lemma 4, provided that T is optimally chosen:[7] imposing (35) together with the condition for optimal switching time, we reobtain (30)-(34) over the finite interval $t \in [0, T)$—see Lemma 6 below. Given this claim of observational equivalence between the cases $T = \infty$ and $0 < T < \infty$, it is possible to make two general remarks regarding phase 1.

The first remark is that, in the resource-based economy, sustained development is not a priori guaranteed. From (31), consumption and output grow at a positive rate if and only if

$$\frac{\psi}{1+\delta} > \rho. \tag{36}$$

For a given discount rate ρ, sustained growth in the first phase requires a moderate degree of resource dependence (low δ) and a sufficiently high productivity of investment (high ψ). If (36) is violated, the negative effect of resource depletion ($\dot{R} < 0$) is stronger than the positive effect of knowledge accumulation ($\dot{A} > 0$), and this implies declining time paths for output and consumption. Inequality (36) may indeed be considered an endogenous-growth variant of the sustainability condition derived in Stiglitz (1974).[8]

The second remark is that the economy exhibits different growth rates in the two phases. More precisely, *the solar-based economy grows faster than the resource-based economy*: from (15) and (31), the growth differential equals

$$\frac{\dot{Y}_2}{Y_2} - \frac{\dot{Y}_1}{Y_1} = \frac{\delta}{\sigma} \cdot \frac{\rho - \psi\,(1-\sigma)}{\sigma\,(1+\delta) - \delta} > 0, \tag{37}$$

where both the numerator and the denominator in the central term are strictly positive by (33) and (34). As may be construed, result (37) is determined by the constraint represented by resource scarcity. While the solar-based economy fully exploits the accumulation of knowledge, the resource-based economy exhibits a lower growth rate because the rate of resource extraction $R(t)$ declines over time. If resources were not essential in the first phase ($\delta = 0$), the two economies would grow at the same, balanced rate $\sigma^{-1}(\psi - \rho)$ determined by knowledge accumulation.

3.3. Optimal Switching Time

The third step of the solution to the social problem is the determination of the optimal timing of backstop technology adoption, T^*. In what follows, we will use a standard terminology. The optimal-timing problem exhibits an *interior solution* if $0 < T^* < \infty$, i.e., the solar-based technology is adopted in finite time but not immediately ($T^* > 0$). The alternatives are represented by the corner solutions

$T^* = 0$ and $T^* = \infty$. In the first case, the optimal policy is that of *immediate adoption*, whereas $T^* = \infty$ represents no adoption—or equivalently, a *permanent delay* in the implementation of the solar-based technology.

The nature of the solution T^* can be clarified as follows. Given the two subproblems of phase 1 and phase 2, denote by $\tilde{C}_1(t; T)$ and $\tilde{C}_2(t; T)$ the time paths of consumption that would be optimal in the two phases for a given switching time T. Using T as an unknown parameter, definitions (8)–(9) imply that the welfare levels associated with the two phases can be expressed as indirect welfare functions that depend on switching time:

$$V_1(T) = \int_0^T U\left(\tilde{C}_1(t; T)\right) e^{-\rho t} dt \quad \text{and} \quad V_2(T) = \int_T^\infty U\left(\tilde{C}_2(t; T)\right) e^{-\rho t} dt. \tag{38}$$

From (38), total present-value welfare can be written as $V(T) = V_1(T) + V_2(T)$. Hence, the optimal timing of technology switching T^* is the instant in which the adoption of the solar-based yields the maximum present-value welfare over the entire time-horizon,

$$T^* = \arg\max_{T \in [0,\infty)} \{V(T) = V_1(T) + V_2(T)\}. \tag{39}$$

Under fairly general conditions, $V(T)$ is defined and finite in T, and differentiable at the switching instant [Seierstad and Sydsaeter (1987)]. If the indirect function $V(T)$ is well behaved—i.e., hump-shaped at least locally—problem (39) exhibits an interior maximum $0 < T^* < \infty$, and the solution is characterized by the first-order condition $dV(T)/dT = 0$; i.e.,

$$\frac{dV_1(T)}{dT} = -\frac{dV_2(T)}{dT}. \tag{40}$$

On one hand, condition (40) represents an intuitive criterion: given a two-phase control problem, the optimal switching time is the instant in which the marginal welfare benefit from increasing the length of one phase equals the marginal welfare cost of reducing the length of the other phase. On the other hand, condition (40) is necessary and sufficient for an optimum only if $V(T)$ is well behaved: because the shape of $V(T)$ is not known a priori, the corner solutions of immediate adoption and permanent delay must be ruled out by showing that (40) is actually associated with a global maximum. To do so, we will implement the following strategy. First, we show that there always exists a unique finite switching instant $T = T' > 0$ that satisfies condition (40). Second, we show that $V(T)$ is strictly concave, implying that $T^* = T'$ is indeed the solution to problem (39).

The behavior of $V(T)$ can be analyzed by applying optimal control theory. A well-known result establishes that, given a control problem with finite initial and terminal dates, the present-value Hamiltonian evaluated in the optimum equals (minus) the derivative of the value function with respect to the (initial) terminal date—see, e.g., Seierstad and Sydsaeter (1987, Theorem 3.9). In the present context, this result is exploited as follows. Denote by $\bar{H}_1(T)$ the Hamiltonian function (18) evaluated at the switching time T along a path satisfying the optimality conditions (19)–(25). Symmetrically, denote by $\bar{H}_2(T)$ the Hamiltonian function (10) evaluated at the switching time T along a path satisfying the optimality conditions (11)–(13). Then the derivatives of the indirect functions $V_1(T)$ and $V_2(T)$ are given by $dV_1(T)/dT = \bar{H}_1(T)$ and $dV_2(T)/dT = -\bar{H}_2(T)$, respectively. As a consequence, the derivative $dV(T)/dT = (dV_1(T)/dT) + (dV_2(T)/dT)$ equals the difference between the two Hamiltonians evaluated at the switching time. We can thus define the gap function

$$\Omega(T) \equiv \bar{H}_1(T) - \bar{H}_2(T) = dV(T)/dT, \qquad \textbf{(41)}$$

and characterize the interior solutions to problem (39) by imposing the condition $\Omega(T) = 0$. The validity of this approach is confirmed by the results of Tomyiama (1985) and Makris (2001), who show that, given a two-stage control problem, an interior solution for the optimal switching time $0 < T^* < \infty$ must satisfy the condition $\bar{H}_1(T^*) = \bar{H}_2(T^*)$.[9] Implementing this procedure yields

LEMMA 5. *There exists a unique finite switching instant $T = T' > 0$ associated with $\Omega(T') = 0$. Function $V(T)$ is strictly concave, and the solution to problem (39) is $T^* = T'$. The optimal switching instant is given by*

$$T^* = \frac{1}{\phi} \ln \left\{ 1 + [1 - (\delta/\sigma)(1-\sigma)]^{\frac{\sigma}{\delta(1-\sigma)}} \beta^{1/\delta} \right\} > 0, \qquad \textbf{(42)}$$

where $\beta \equiv (nS_0\phi)^{\delta} N^{\gamma-\delta} \alpha^{-1} (mG)^{-\gamma}$. If preferences are logarithmic, $\sigma = 1$, the same results hold with an optimal switching time $T^* = (1/\phi)\ln[1 + (\beta^{1/\delta}/e)]$.

Lemma 5 is a crucial result of this paper. It shows that, given the assumptions made so far, the optimal timing of backstop technology adoption is unique, and can be expressed as a function of the model parameters. Expression (42) shows that the optimal switching time is inversely related to the speed of resource depletion: the higher ϕ, the sooner the solar-based technology is adopted. The role of the other parameters is generally ambiguous, instead, because it depends on whether σ is above or below unity. A numerical example that confirms Lemma 5 is described in Figure 1, where the indirect welfare function $V(T)$ and the gap function $\Omega(T)$ are obtained for a given set of parameter values.[10] The indirect welfare function achieves a maximum in $T^* = 16.3$, associated with the horizontal intercept of the gap function, $\Omega(T^*) = 0$.

We now have all the elements to characterize the optimal path of the economy in both phases. The following section describes the main results of the analysis, and discusses the economic consequences of backstop technology adoption.

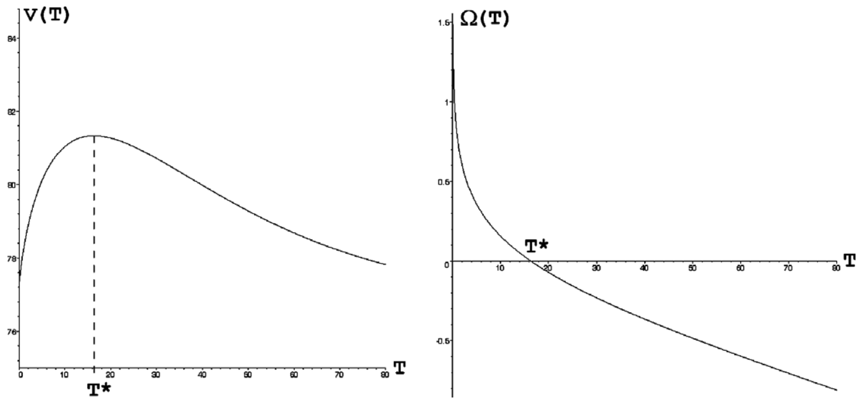

FIGURE 1. A numerical example of indirect welfare function and optimal switching time. The left graph depicts the welfare-timing relationship $V(T) = V_1(T) + V_2(T)$. The optimal switching time T^* corresponds to the horizontal intercept of the gap function $\Omega(T)$, depicted in the right graph.

4. CONSEQUENCES OF TECHNOLOGY SWITCHING

On the basis of our previous results, the optimal path of the economy over the whole time horizon $t \in [0, \infty)$ can be described as follows. In the second phase, delimited by the interval $t \in (T^*, \infty)$, the optimal path is characterized by the conditions already stated in Lemma 1. In the first phase, delimited by the interval $t \in [0, T^*)$, the economy exploits exhaustible resources according to technology (1). The following lemma establishes that, during phase 1, the economy exhibits balanced growth in each instant, and displays the same properties, in terms of growth rates and consumption/saving propensities, of the infinite-horizon resource-based economy described in Lemma 4:

LEMMA 6. *Given an optimal switching time $T = T^*$, the resource-based economy follows an optimal path in which the optimal propensity to consume equals (30), output and consumption grow at the constant rate (31), and knowledge grows at the constant rate (32) in each $t \in [0, T^*)$. The optimal path is well defined if and only if parameters satisfy (33)–(34).*

Lemma 6 and equation (37) imply that, along the optimal path, the resource-based economy grows at slower rate with respect to the solar-based economy. As noted before, the reason is that the growth process in phase 1 is constrained by resource scarcity. The most interesting aspect is related to the immediate consequences of adopting the backstop technology:

PROPOSITION 7. *The adoption of the solar-based technology implies discrete jumps in consumption, output and growth. The transition to the solar-based economy is characterized by sudden reductions in consumption and output* *levels,*

$$C_1\left(T^*\right)/C_2\left(T^*\right) = \left\{\sigma\left[\sigma - \delta\left(1-\sigma\right)\right]^{-1}\right\}^{\frac{1}{1-\sigma}} > 1,$$
$$Y_1\left(T^*\right)/Y_2\left(T^*\right) = \left\{\sigma\left[\sigma - \delta\left(1-\sigma\right)\right]^{-1}\right\}^{\frac{\sigma}{1-\sigma}} > 1,$$

and, from (37), an immediate increase in the growth rate. If preferences are not logarithmic, $\sigma \neq 1$, technology adoption also implies a discrete jump in consumption/saving propensities,

$$c_1\left(T^*\right)/c_2\left(T^*\right) = \sigma\left[\sigma - \delta\left(1-\sigma\right)\right]^{-1},$$

where $\sigma \lesseqgtr 1$ implies $c_1\left(T^*\right) \gtreqless c_2\left(T^*\right)$.

Proposition 7 contains the main results of this paper. The adoption of the backstop technology involves an immediate lowering of consumption and output, whereas the sign of the adjustments in consumption propensities and saving rates depends on the elasticity of intertemporal substitution. The next sections discuss the scope and economic interpretation of these results, including additional remarks on the behavior of marginal productivities (Section 4.3) and a numerical simulation (Section 4.4).

4.1. Optimal Jumps in Consumption Levels

The sudden reduction in consumption associated with backstop technology adoption is intimately linked to the existence of a growth differential between the two phases. Lemma 6 and Proposition 7 show that the optimal transition to the new technology is characterized by a precise trade-off between growth and level effects: the solar-based economy grows faster than the resource-based economy, but the adoption of the backstop technology induces a sudden reduction in consumption and output. The underlying reason is as follows. On one hand, technology switching implies a transition from slow to fast growth because the resource-based economy is constrained by natural scarcity, whereas the solar-based economy fully exploits the growth potential of R&D investment. On the other hand, because the productivity of knowledge accumulation changes between the two phases, there is an intertemporal reallocation effect whereby consumption levels are reduced by technology switching.[11] During phase 1, the exhaustible resource is exploited to obtain high consumption levels in spite of slower growth. When the economy switches to solar-based energy, consumption levels are lower for but this is optimal, because the negative level effect of technology switching is compensated—in terms of present-value welfare—by the higher growth rate that the economy enjoys from $t = T^*$ onward.

The presence of discrete jumps in consumption is a novel feature with respect to the recent endogenous-growth literature: the vast majority of sustainability models postulate a continuous process of resource-augmenting technical progress whereby resource inputs are progressively substituted for by knowledge-type capital [Barbier (1999)] or expanding varieties of intermediate products [Scholz and Ziemes (1999)]. Our results show that the introduction of a backstop technology in conjunction with a linear accumulation law of knowledge involves nonsmooth dynamics with a precise characteristic: due to the positive growth differential between the two phases, switching to the new technology is optimal even though the resulting current benefits in terms of consumption are lower than those generated by traditional technologies. This result appears relevant from a policy-making perspective, because nontraditional energy sources such as wind and solar power are regarded as less productive now but are more likely to guarantee sustainable growth in the future.

4.2. Optimal Jumps in Consumption Propensities

Proposition 7 shows that although the sign of level and growth effects is unambiguous, the way in which consumption and saving propensities adjust to the new technology depends on the elasticity of intertemporal substitution. If $\sigma < 1$ we have $c_1^* > c_2^*$; i.e., the propensity to consume is suddenly reduced by the adoption of the backstop technology. The opposite phenomenon arises when $\sigma > 1$, which implies an upward jump in the consumption propensity, $c_1^* < c_2^*$. When preferences are logarithmic, the adoption of the backstop technology has no effect on the saving rate. Nonetheless, there are discrete jumps in consumption, output, and growth rates: when $\sigma = 1$, the size of the reduction in consumption levels is $C_1(T^*)/C_2(T^*) = e^\delta$, so that the magnitude of the level effects of technology adoption increases exponentially with the degree of resource dependence.[12] These results can be interpreted as standard effects of intertemporal substitution: switching from phase 1 to phase 2 involves a transition from low to high growth, and the behavior of consumption propensities in the transition is in fact equivalent to that arising in life-cycle models when consumers adjust their saving rates in response to an increase of the rate of return. When $\sigma < 1 \quad (\sigma > 1)$, the transition to a higher rate of return induces a lower (higher) propensity to consume because the willingness to postpone consumption dominates (is dominated by) the willingness to smooth the consumption profile.

4.3. Marginal Productivities

The behavior of marginal productivities along the optimal path deserves some comment. In the present model, the social profitability rates associated with the primary energy sources are represented by $\partial Y_1/\partial R$ and $\partial Y_2/\partial G$. As shown in the Appendix, the optimal switching time is characterized by

$$\frac{\partial Y_1\left(T^*\right)/\partial R\left(T^*\right)}{\partial Y_2\left(T^*\right)/\partial G} = \left(\frac{n^\delta N^{\gamma-\delta}}{\alpha m^\gamma}\right)^{1/\delta} \frac{\delta G^{1-\frac{\gamma}{\delta}}}{\gamma} \left\{\frac{\sigma - \delta\left(1-\sigma\right)}{\sigma}\right\}^{\frac{\sigma(1-\delta)}{\delta(1-\sigma)}}. \qquad \textbf{(43)}$$

Expression (43) shows that the marginal productivities of primary energy sources are generally different at time T^*.[13] This result allows us to draw some distinctions and similarities with the literature on backstop technology adoption. The early models pioneered by Hoel (1978), Dasgupta et al. (1982), and Dasgupta and Stiglitz (1981) emphasize the fact that technology switching occurs when resource-based energy has become as expensive as the backstop substitute. This conclusion hinges on the assumption that current prices reflect current marginal costs at each point in time. In the present model, the optimal switching time is determined in a different context: the economy is centralized and the costs and benefits of adoption are represented by the present-value streams of utilities enjoyed during the two phases. Condition (40) equates marginal social benefits and costs and the value functions V_1 and V_2 incorporate all the aggregate effects that characterize the two phases of economic development—in particular, the fact that the economy develops at slower rates when production possibilities are constrained by resource scarcity (phase 1), whereas solar-based technologies guarantee sustained and faster growth (phase 2). The shadow prices μ_1 and μ_2 reflect the social value of accumulating knowledge and incorporate the expected growth effects of alternative technologies. Hence, the gap between current marginal productivities in (43) is due to the peculiarities of the present model, i.e., the use of a welfare-based criterion in a macroeconomic setting and the existence of an endogenous growth process. Different profitability rates may nonetheless be obtained in partial equilibrium models for other reasons: in a competitive framework, new technologies may be adopted even if they are more expensive than old ones when current marginal costs differ from the effective opportunity cost of adoption. This may happen (a) when public intervention internalizes the negative pollution externalities of resource-based energy [Tahvonen (1997)] or (b) because firms are forward-looking and take into account future cost reductions due to learning by doing [Braumollé and Olson (2005)], or for both reasons at the same time [Chakravorty et al. (2009)]. The fact that the social planner anticipates the growth effects of solar-based energy in the present model may play a role similar to that of the internalization of future cost reductions due to learning by doing in Tsur and Zemel (2003). In this respect, our results are new because Tsur and Zemel (2003) study firms' optimal R&D in isolation from endogenous growth and macroeconomic issues. More generally, because all the aforementioned contributions employ partial equilibrium frameworks, our results concerning the optimal jumps in consumption levels, growth rates, and saving propensities are peculiar features of the present model.

4.4. Numerical Simulations

The results stated in Proposition 7 can easily be verified by numerical simulations. Two examples are reported in Table 1, and graphically described in Figure 2. Except for σ, the parameter values are the same as used in Figure 1; see note 10. With $\sigma = 0.8 < 1$, we obtain an optimal switching time $T^* = 16.2$ and a downward jump in the consumption propensity associated with the adoption

TABLE 1. Optimal jumps: Simulation results

	$\sigma = 0.8$		$\sigma = 1.2$	
Output in T^*	$Y_1 = 37.4$	$Y_2 = 28.9$	$Y_1 = 34.5$	$Y_2 = 27.0$
Cons. propensities	$c_1^* = 0.62$	$c_2^* = 0.58$	$c_1^* = 0.69$	$c_2^* = 0.72$
Consumption in T^*	$C_1 = 23.2$	$C_2 = 16.8$	$C_1 = 23.9$	$C_2 = 19.5$
Switching time	$T^* = 16.2$		$T^* = 16.4$	

Notes: See also Figure 2.

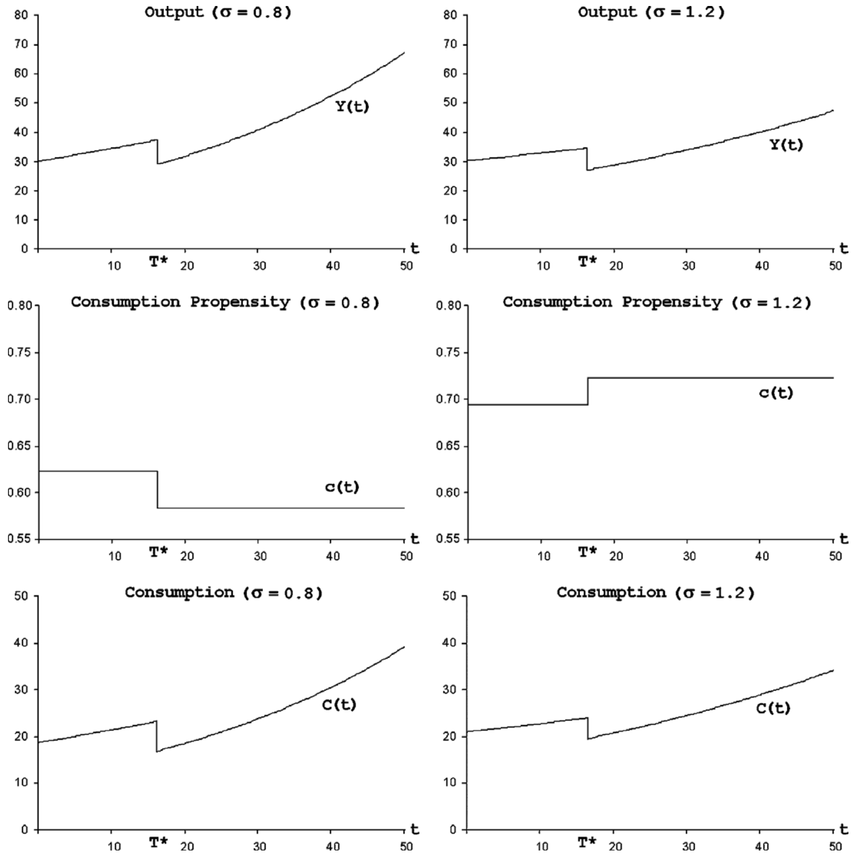

FIGURE 2. Optimal paths of output, consumption, and consumption propensities for the two cases $\sigma = 0.8$ and $\sigma = 1.2$. See Table 1 for details and note 10 for the list of parameter values.

of the solar-based technology. The growth rate switches from $\dot{Y}_1/Y_1 = 1.3\%$ to $\dot{Y}_2/Y_2 = 2.5\%$. With $\sigma = 1.2 < 1$, technology adoption is slightly delayed and implies a sudden increase in the propensity to consume. The growth rate switches from $\dot{Y}_1/Y_1 = 0.8\%$ to $\dot{Y}_2/Y_2 = 1.6\%$.

TABLE 2. Growth gains and consumption losses: The role of resource dependence

Resource share	Consumption propensity		Growth gain	Consumption gap
$\delta = 0.10$	$c_1^* = 0.71$	$c_2^* = 0.76$	$+0.30\%$	-3.81%
$\delta = 0.17$	$c_1^* = 0.68$	$c_2^* = 0.76$	$+0.49\%$	-6.27%
$\delta = 0.25$	$c_1^* = 0.65$	$c_2^* = 0.76$	$+0.69\%$	-8.90%

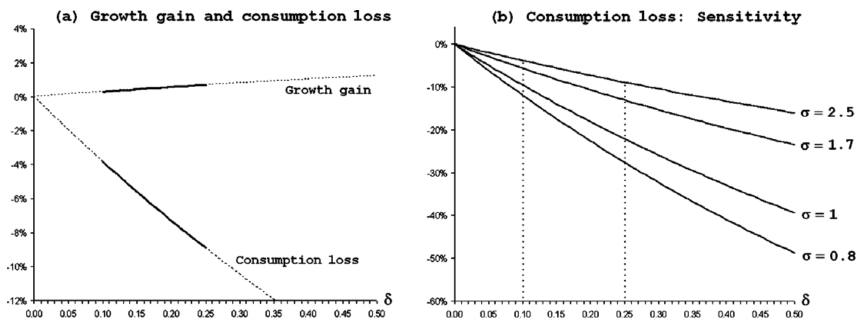

FIGURE 3. (a) The growth gain (37) and the percentage consumption loss as functions of the resource share δ. (b) The consumption loss becomes more elastic to δ as σ decreases.

The quantitative relevance of the trade-off between growth and level effects is crucially determined by the value of the resource share in production: both the growth differential between the two phases and the size of the associated consumption jump are increasing functions of δ. The relevance of the consumption shock can be assessed in relation to the growth gain yielded by solar-based technologies under the baseline parameters used in standard simulations of endogenous-growth models. Following Alvarez-Cuadrado (2008), we set $\rho = 0.04$ and $\psi = 0.105$, with an intertemporal elasticity of substitution $\sigma = 2.5$ given by the estimates of Ogaki and Reinhart (1998). In capital-resource models, the conventional wisdom is that the value of δ lies between 0.10 and 0.25, Because this is complementary to the standard estimates of capital and labor shares. In the interval $\delta \in (0.10, 0.25)$, these parameters imply that the consumption propensity in phase 1 ranges from 0.71 to 0.65. The growth rate ranges from 2.3% to 1.9% in phase 1, and the associated growth rate in phase 2 is 2.6%. The trade-off between growth and level effects is represented as a function of δ in Figure 3a. The increasing curve is the *growth gain* of technology adoption, measured by expression (37). The decreasing curve is the *consumption loss*, measured by the percentage gap $[C_2(T^*)/C_1(T^*)]-1$. The bold line emphasizes the relevant interval $\delta \in (0.10, 0.25)$. Figure 3a suggests that the consumption loss is highly sensitive to resource dependence. The consequences of technology switching are summarized in Table 2.

Looking at the extreme cases, strong resource dependence ($\delta = 0.25$) implies a consumption loss of 8.9%, that is, more than two times the consumption loss associated with weak dependence ($\delta = 0.10$). Stronger resource dependence also depresses growth in phase 1 without affecting growth in phase 2, which implies bigger growth differentials. These results suggest a possible extension of the analysis: if the resource share can be modified by dedicated technical change, i.e., resource-saving technological progress, the society may substantially relieve the consumption shock by investing in δ-reducing projects before the resource-based technology is completely abandoned in favor of the solar-based technology. It is possible to imagine situations in which similar smoothing strategies would be optimal, e.g., when consumption shocks beyond a certain size bear negative side effects at the macroeconomic level. This point is furthermore relevant in view of the robustness of the sensitivity result: if we set σ below the estimated value 2.5, the sensitivity of the consumption loss to the resource share increases further, as shown in Figure 3b.

5. ALTERNATIVE ASSUMPTIONS

5.1. Transitional Dynamics

The existence of closed-form solutions to the optimal timing problem hinges on the absence of transitional dynamics. It may be argued that, in more general settings, the main results still hold, provided that a positive growth differential between phase 2 and phase 1 arises and that knowledge accumulation is proportional to saving rates at least asymptotically. In fact, if growth rates are not constant but balanced growth is asymptotically achieved, our previous reasoning applies to average growth rates: despite the absence of closed-form solutions for consumption paths, the optimal switching time is still determined by condition (40), where V_1 and V_2 incorporate the cumulative effects of the sequence of growth rates obtained within each phase. This suggests that a differential in average growth rates in favor of the backstop technology would imply adoption in finite time. Beyond this general remark, the absence of transitional dynamics in our model is mainly due to the assumptions of (i) constant flow of solar-based energy and (ii) linearity in the knowledge transition law (4). Each point is briefly discussed below.

Renewable resources. In the model presented, the constant flow G represents a renewable resource that is fully renewed within each instant (e.g., solar). Alternatively, we can assume that the backstop technology exploits a different resource stock displaying a partial regeneration process. In this scenario, $G(t)$ becomes a time-varying flow (e.g., biofuel) linked to the stock $Q(t)$ of a primary resource (e.g., sugar cane) by a harvesting law of the type

$$\dot{Q}(t) = \pi(t) Q(t) - G(t),$$

where $\pi(t)$ is the marginal rate of natural regeneration. The second-phase problem would thus include an additional state variable. If $\pi(t)$ is constant, however, the optimal path in phase 2 still does not display transitional dynamics: the marginal rate of resource renewal adds linearly to the rate of productivity growth in the Keynes–Ramsey rule [Valente (2005)] and its time-invariance yields a constant growth rate in consumption along the entire optimal path if the rate of return to investment is constant as well [Valente (2008); Section 4]. In this case, results do not change, because the growth rate of phase 1 will surely be lower than that in phase 2. Transitional dynamics due to natural regeneration only arises when $\pi(t)$ is time-varying—e.g., because regeneration is stock-dependent, as in logistic bioeconomic models. In this scenario, the growth differential between phases 1 and 2 varies over time but remains positive in the long run, provided that $\lim_{t\to\infty} \pi(t) > 0$ holds along the optimal path. The model can be extended further by assuming that the resource used in phase 1 is also partially renewable, but its average renewal rate is lower than that of the backstop substitute. In this case, the growth differential is still in favor of the backstop substitute and transitional dynamics does not arise if both marginal regeneration rates are constant.

Nonlinear transition laws. Equation (4) postulates that knowledge growth is proportional to saving rates in each point in time. This assumption can be microfounded in several ways by considering a competitive economy in which the R&D sector comprises many firms and productivity growth is linked to the average level of R&D expenditures across the economy through knowledge spillovers [e.g., Valente (2008)]. Alternatively, it is possible to replace (4) by another function of the form $\dot{A}/A = \bar{\varphi}(Y_i(t), D_i(t), d_i(t))$, where the growth rate of the knowledge stock is positively linked to aggregate investment and negatively linked to output levels due to congestion effects that scale down productivity [Acemoglu (2002)] or to sectoral spillovers in entry [Peretto (1998)]. In these more general cases, knowledge growth is not in fixed proportion with the saving rate in each point in time but becomes linear in saving rates asymptotically as the economy approaches balanced growth. In the present setting, transitional dynamics in $\dot{A}/A$ would arise in both phases, with the optimal switching time being determined by the cumulative welfare effects of the sequence of growth rates. As discussed above, it may be argued that the existence of a differential in average growth rates in favor of the backstop technology would still imply adoption in finite time.

5.2. Switching Costs

Another assumption of the model is that the adoption of the backstop technology does not involve any extra cost. The reason is that our analysis concentrates on the choice of the optimal adoption timing given that the backstop technology is already available at time $t = 0$, the implicit assumption being that R&D efforts have already been undertaken prior to time zero. It is nonetheless possible to introduce switching costs occurring at time T, representing, e.g., fixed costs of industry conversion. In a two-stage problem, Makris (2001) introduces switching costs by augmenting the social objective function by an external term:

$$V(T) = V_1(T) + V_2(T) - e^{-\rho T}\Lambda(T). \tag{44}$$

Note that, in the present model, $\Lambda(T)$ should be interpreted as an instantaneous welfare loss suffered at time T. The consequences for optimality conditions are twofold. First, the optimal switching time would be characterized by $dV(T)/dT = 0$, i.e.,

$$\frac{dV_1(T)}{dT} = -\frac{dV_2(T)}{dT} + \frac{d}{dT}\left[e^{-\rho T}\Lambda(T)\right], \tag{45}$$

which says that the marginal benefit of delaying adoption, $dV_1(T)/dT$, must equal the *effective* marginal cost of delaying adoption represented by the direct cost of reducing the length of the second phase, $-dV_2(T)/dT$, plus the increase in the present-value of the switching cost, i.e., the last term in (45). Second, if the switching cost is a function of the knowledge stock, $\Lambda(T) = \Lambda(A(T))$, condition (20) is replaced by [see Makris (2001)]

$$\lim_{t\to T^-}\mu_1(t) = \lim_{t\to T^+}\mu_2(t) - e^{-\rho T}\frac{d\Lambda(A(T))}{dA(T)}. \tag{46}$$

The right-hand side of (46) shows that the *effective* marginal benefit of knowledge accumulation at the beginning of phase 2 is now represented by the instantaneous marginal benefit $\mu_2(T)$ plus the present value of the marginal reduction in the switching cost due to knowledge accumulation, i.e., the last term in (46). The bottom line is that, on the one hand, the existence of switching costs modifies the timing of adoption through (45). On the other hand, from (46), a knowledge-dependent cost modifies the relative saving rates between the two phases. For example, applying condition (46) to the present model with logarithmic preferences, we obtain (see the Appendix)

$$\frac{1}{c_2^*} - \frac{1}{c_1^*} = \frac{d(\Lambda(A(T^*)))}{dA(T^*)}\psi A(T^*). \tag{47}$$

Equation (47) shows that if knowledge implies a reduction in the switching cost, the consumption propensity is lower in the first phase: $\partial\Lambda(A)/\partial A < 0$ implies $c_1^* < c_2^*$. The intuition is that the society should save a larger fraction of output during phase 1 because a larger knowledge stock at time T^* guarantees lower switching costs. This suggests that switching costs may reduce the size of the consumption shock occurring at time T^* because they contrast with the welfare gains associated with faster growth in phase 2 and thus represent a *ceteris paribus* reduction of the benefits of adoption. Pursuing this argument and assessing the size of the consumption shock, however, requires further analysis, because formulation (44) does not specify the important trade-off that arises between "real" switching costs—e.g., foregone output—and the consumption level at the beginning of phase 2.[14]

6. CONCLUSIONS

This paper has studied the optimal timing of backstop technology adoption in a two-phase endogenous growth model where production requires energy initially obtained from exhaustible resources. A backstop technology represented by solar-based energy is available, and a benevolent social planner decides whether and when to abandon traditional oil-based energy in favor of the new technology. The model exhibits closed-form solutions for the optimal switching time and the time paths of all endogenous variables. The transition to solar-based energy involves discrete jumps in consumption and output: the adoption of the backstop technology implies a sudden reduction in consumption and output levels, an increase in the growth rate, and instantaneous adjsutments in the saving propensity. The intuition for these results is as follows. On the one hand, technology switching implies a transition from slow to fast growth because the resource-based economy is constrained by natural scarcity whereas the solar-based economy fully exploits the growth potential of R&D investment. On the other hand, because the productivity of knowledge accumulation changes between the two phases, there is an intertemporal reallocation effect whereby consumption levels are reduced by technology switching. During the first phase, the traditional technology yields slow growth and the exhaustible resource is exploited to obtain high consumption levels in the short run. When the economy switches to solar-based energy, consumption levels are lower, but this is optimal, because the negative level effect of technology switching is compensated—in terms of present-value welfare—by the higher growth rate that the economy enjoys from the instant of adoption onward. Hence, due to the positive growth effects of technology switching, the adoption of new solar-based techniques is optimal even though the associated current benefits in terms of consumption are substantially lower. This result appears relevant from a policy-making perspective because nontraditional energy sources such as wind and solar power are regarded as less productive now but are more likely to guarantee sustainable growth in the future.

The analysis shows that welfare-based criteria in conjunction with endogenous-growth mechanisms yield new results with respect to the partial-equilibrium literature on backstop technologies: the optimal switching time is determined by a more complete forward-looking criterion that takes into account the future growth effects of technology adoption and the behavior of saving rates. The structure of the model suggests a number of extensions that can be implemented in this framework. A natural question relates to the effects of market failures on the optimal timing of structural change. Endogenous growth models typically assume that non-decreasing returns hinge on the presence of externalities. In this framework, decentralized competitive equilibria are characterized by intertemporal allocations that differ from the social optimum analyzed here. Studying the nature and consequences of optimal policies when these externalities also affect the timing of backstop technology adoption is the main suggestion for future research.

NOTES

1. In the context of exhaustible resources, the sustainability condition derived by Stiglitz (1974) establishes that nondeclining consumption in the long run requires the utility discount rate not be less than the rate of resource-saving technical progress. The same condition is valid in endogenous-growth models where both the speed and the direction of technical progress are endogenous—see Di Maria and Valente (2008). If the natural resource is renewable, the Stiglitz (1974) condition is augmented by the marginal rate of resource regeneration—see Valente (2005) for a generalization of the neoclassical framework—and can be expressed, in an endogenous-growth setting, in terms of the rate of resource use—see Aznar-Marquez and Ruiz-Tamarit (2005).

2. Barro and Sala-i-Martin (2004, p. 300) show that linear accumulation laws eliminate scale effects in standard expanding-varieties models *à la* Rivera-Batiz and Romer (1991)—see also Acemoglu (2002, Section 4)—by making the equilibrium growth rate of output independent of the population size. Explicit micro-foundations of (4) are provided by decentralized models with mixed vertical and horizontal innovations [Peretto (1998)] or learning processes with standard Lucas-type knowledge spillovers [Valente (2008)]. Barro and Sala-i-Martin (2004, p. 302) also note that linear accumulation laws fit the data better because, in most industrialized countries, the growth rate of productivity appears to be positively related to the ratio between R&D expenditures and output with a proportionality coefficient that is relatively stable over time.

3. All lemmas and propositions are proved in the Appendix.

4. By integration of the accumulation law (4), we have $A(t) = A(T)e^{\psi d_2^*(t-T)}$, where $d_2^* \equiv 1 - c_2^*$ implies that $A(t) = A(T)e^{(1/\sigma)(\psi-\rho)(t-T)}$. Substituting this result into $C_2(t) = c_2^* Y_2(t) = c_2^* \alpha A(t)(mG)^\gamma N^{1-\gamma}$, we obtain equation (17) in the text.

5. For simplicity, the analysis abstracts from possible corner solutions where the nonnegativity constraint $R(t) \geq 0$ is binding. In the present setting, corner solutions with $R(t) = 0$ can be safely neglected because, due to the assumption of Cobb–Douglas technology in (1), the resource is essential for production [$R(t) = 0$ implies $Y_1(t) = 0$] and the Inada condition $\lim_{R\to\infty}(\partial Y_1/\partial R) = 0$ holds. Satisfying these conditions is sufficient for an interior solution with $R(t) > 0$ in each t along the optimal path: see Dasgupta and Heal (1974, Proposition 5).

6. As shown in Lemmas 4 and 6, optimality requires a declining rate of resource over time, so that parameters must satisfy the restriction $\phi > 0$.

7. The only situation in which c_1^* does not have to be determined simultaneously with the optimal switching time $T = T^*$ is the case of logarithmic preferences, $\sigma = 1$. In fact, when $U(C_i) = \ln C_i$, the terminal condition (35) implies $c_1^*/c_2^* = 1$ independently of the switching time T. In this case, we have $c_2^* = \rho/\psi$ from (14), and therefore equal propensities $c_1^* = c_2^* = \rho/\psi$ in both phases independently of the switching time T.

8. Assuming capital-resource technology of the type $Y = AK^{1-\delta}R^\delta$ with an exogenous rate of Hicks-neutral technical progress $\dot{A}/A = \nu$, Stiglitz (1974) showed that output and consumption are asymptotically increasing if $\nu/\delta > \rho$, where δ is the resource share in production.

9. The social problem satisfies all the hypothesis of Theorem 1 in Makris (2001) with zero switching costs [$\Phi = 0$ in Makris's (2001) notation]. The condition $\bar{H}_1(T^*) = \bar{H}_2(T^*)$ follows directly from equation [15] in Makris (2001, p. 1939). See also Tomiyama (1985, Theorem 1).

10. Parameter values are $\alpha = 2$, $m = n = G = 1$, $\gamma = 0.3$, $S_0 = 1000$, $\delta = 0.25$, $\psi = 0.06$, $\rho = 0.04$, $\sigma = 1$, $A_0 = 10$. Notice that, in the logarithmic case $\sigma = 1$, the indirect welfare function $V(T)$ can be computed in an easy manner because, as shown in note 7, the optimal consumption propensity in the resource-based economy equals $c_1^* = \rho/\psi$ in each $t \in [0, T)$ independent of the value of T. This implies that we have simple closed-form solutions for the conditional consumption path $\tilde{C}_1(t; T)$ for any value of T, and this allows us to obtain explicit expressions for the indirect welfare subfunctions $V_1(T)$ and $V_2(T)$.

11. The intertemporal reallocation effect is particularly evident under logarithmic preferences: when $\sigma = 1$, the rate of knowledge accumulation is identical in the two phases because the saving rate is not affected by technology switching, but the growth rate in phase 2 is higher, whereas the consumption level is immediately reduced.

12. As shown in the Appendix—see equation (A26)—the optimal switching time is characterized by the condition $\ln(C_1(T^*)/C_2(T^*)) = \delta$.

13. Even assuming a convenient set of parameters—e.g., identical production shares $\gamma = \delta$ and unit productivity indices $\alpha = m = n = 1$—the right-hand side of (43) differs from unity. This is due to the term in curly brackets, which indeed determines the size of consumption and output jumps (cf. Proposition 7).

14. Specification (44) represents switching costs in the form of an external welfare loss $\Lambda(T)$. A more plausible specification is the following. Suppose that the switching cost is represented by χ units of foregone output at instant T. The budget constraint at time T must be modified as $Y_2(T) = C(T) + D(T) + \chi(T)$. In this case, the welfare loss associated with the switching cost is already internalized in the objective function through the value of instantaneous utility at switching instant, $U(C_2(T)) = U(Y_2(T) - D_2(T) - \chi(T))$. Because the objective function remains $V(T) = V_1(T) + V_2(T)$, the resulting optimality conditions will generally differ from (45) and (46), which refer to specification (44). But the analysis will also differ from that in Section 3 because $c_2 = 1 - d_2$ does not hold anymore at time T. This specification guarantees that the consumption jump at time T will take into account the simultaneous output loss generated by switching costs.

15. Substituting $\hat{C}_1 = \hat{c}_1 + \hat{Y}_1 = \hat{c}_1 + \hat{A} + \delta\hat{R} = \hat{c}_1 + \psi(1 - c_1) + \delta\hat{R}$ in (26), and plugging (27) into the resulting expression, we obtain (A.13).

16. In Lemma 4, the restrictions (33) and (34) are associated with an infinite switching time $T = \infty$. As shown in the subsequent Lemma 6, if we set the switching instant equal to $T = T'$, the resource-based economy exhibits the same properties listed in Lemma 4.

17. The result that c_1 coincides with optimal propensity c_1^* obtained for the case $T = \infty$ in Lemma 4 may suggest conjecturing that c_1^* equals $\bar{c}_1$ in each period independently of the value of switching time T. This conjecture is wrong: it can be shown that, when the elasticity of intertemporal substitution differs from unity, $\sigma \neq 1$, we have $c_1^* = \bar{c}_1$ for $T = \infty$, $c_1^* = \bar{c}_1$ for $T = T^*$, but $c_1^* \neq \bar{c}_1$ for other finite values of $T \neq T^*$. The only case in which the optimal propensity in phase 1 is independent of the switching time T arises when preferences are logarithmic: as shown in note 7, setting $\sigma = 1$ implies $c_1^* = c_2^* = \rho/\psi$.

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APPENDIX A

A.1. DERIVATION OF (11)–(13)

From the aggregate constraint of the economy and technology (2), consumption equals $C_2 = (1 - d_2)Y_2 = (1 - d_2)\alpha A(mG)^\gamma N^{1-\gamma}$. The second-phase problem can be thus specified as

$$\max_{\{d_2\}_{t=T}^{\infty}} V_2 = \int_T^{\infty} U\left[(1-d_2)\,\alpha A\,(mG)^{\gamma}\,N^{1-\gamma}\right] e^{-\rho t} dt$$

subject to $\dot{A} = \psi A d_2$, with $A(T)$ given and the saving propensity d_2 acting as control variable. The Hamiltonian (10) can be written as

$$H_2 = U\left[(1-d_2)\,\alpha A\,(mG)^{\gamma}\,N^{1-\gamma}\right] e^{-\rho t} + \mu_2 \psi A d_2. \tag{A.1}$$

Equation (11) is obtained from $\partial H_2/\partial d_2 = 0$, condition (12) is the co-state equation $\partial H_2/\partial A = -\dot{\mu}_2$, and (13) is the standard transversality condition.

A.2. PROOF OF LEMMA 1

Denoting by $\hat{x} \equiv \dot{x}/x$ the instantaneous growth rate of the generic variable $x(t)$, time-differentiation of (11) yields

$$\hat{\mu} = \hat{Y}_2 - \hat{A} + \hat{U}' - \rho = \hat{U}' - \rho, \tag{A.2}$$

where the last term comes from the fact that $\hat{Y}_2 = \hat{A}$. Plugging (11) into (12), we obtain

$$\hat{\mu}_2 = -\psi. \tag{A.3}$$

Combining (A3) with (A2) and $\hat{U}' = -\sigma \hat{C}_2$, we obtain

$$\hat{C}_2 = \sigma^{-1}(\psi - \rho). \tag{A.4}$$

Substituting $\hat{C}_2 = \hat{c}_2 + \hat{Y}_2$ and $\hat{Y}_2 = \hat{A} = \psi d_2$ in (A4), and using $c_2 = 1 - d_2$, the optimal propensity to consume must satisfy

$$\hat{c}_2 = \psi c_2 + \sigma^{-1}(\psi - \rho) - \psi, \tag{A.5}$$

Relation (A5) is globally unstable with a unique fixed point

$$\bar{c}_2 = 1 - \frac{\psi - \rho}{\psi \sigma}. \tag{A.6}$$

Explosive dynamics of c_2 can be ruled out, as they lead to either negative consumption or negative output in finite time. The optimal propensity c_2^* is thus equal to $\bar{c}_2$ in each $t \in [T, \infty)$, which proves equation (14). A constant propensity to consume implies $\dot{Y}_2/Y_2 = \dot{C}_2/C_2 = \dot{A}/A$. Imposing $0 < c_2^* < 1$, we obtain restriction (16), which completes the proof. ■

A.3. PROOF OF LEMMA 2

If $T \rightarrow \infty$, the first-phase problem is a standard infinite-horizon optimal control and the transversality condition on the knowledge stock is (19). If solar energy is adopted at some finite T, instead, the social problem belongs to the class of two-stage problems analyzed in Tomiyama (1985) and Makris (2001). Using the current notation, expression (20) corresponds to the optimality condition derived, e.g., in Tomiyama (1985, Theorem 1, equation [15]). ■

A.4. PROOF OF LEMMA 3

From (1), consumption in phase 1 equals $C_1 = (1 - d_1)Y_2 = (1 - d_1)A(nR)^{\delta}N^{1-\delta}$. The first-phase problem can be thus specified as

$$\max_{\{d_1,R\}_{t=0}^{T}} V_1 = \int_0^T U\left[(1-d_1)\,A\,(nR)^{\delta}\,N^{1-\delta}\right]e^{-\rho t}dt$$

subject to $\dot{A} = \psi A d_1$, with $A(0) = A_0$ given, and to $\dot{S} = -R$, with $S(0) = S_0$ given. The saving propensity d_1 and the rate of resource use R act as control variables. The present-value Hamiltonian (18) can be written as

$$H_1 = U\left[(1-d_1)\,A\,(nR)^{\delta}\,N^{1-\delta}\right]e^{-\rho t} + \mu_1\psi A d_1 - \lambda R. \tag{A.7}$$

Equations (21) and (22) are given by the first-order conditions $\partial H_1/\partial d_1 = 0$ and $\partial H_1/\partial R = 0$, respectively. The co-state condition $\partial H_1/\partial A = -\dot{\mu}_1$ yields

$$-\dot{\mu}_1 = (1-d_1)\cdot U'(C_1)\cdot e^{-\rho t}\,(Y_1/A) + \mu_1\psi d_1, \tag{A.8}$$

where we can substitute (21) to obtain (23). Equation (24) is given by the co-state condition $\partial H_1/\partial S = -\dot{\lambda}$. The transversality condition on the resource stock reads

$$\lambda(T)\,S(T) = 0. \tag{A.9}$$

Because $\lambda(t)$ is constant by (24), condition (A9) requires exhausting the whole resource stock at the end of the first phase, $S(T) = 0$. Integration of the dynamic law (5) between $t = 0$ and $t = T^*$ yields

$$S(T) = S_0 - \int_0^T R(t)\,dt.$$

Substituting $S(T) = 0$ into the above expression yields equation (25). ■

A.5. DERIVATION OF (26)–(27)

Because $\dot{\lambda}(t) = 0$ by (24), time-differentiation of (22) yields

$$\hat{R} = \hat{U}' + \hat{c}_1 + \hat{Y}_1 - \rho = \hat{U}' + \hat{C}_1 - \rho. \tag{A.10}$$

Time-differentiating (21), and eliminating $\hat{\mu}_1$ by means of (23), we have

$$\hat{U}' = \rho - \psi - \hat{Y}_1 + \hat{A}. \tag{A.11}$$

Time-differentiating (1), we have $\hat{Y}_1 = \hat{A} + \delta\hat{R}$. Plugging this result into (A.11), and substituting (A.10), we obtain

$$\hat{U}'(1+\delta) = \rho - \psi - \delta\hat{C}_1 + \delta\rho. \tag{A.12}$$

Substituting $\hat{U}' = -\sigma\hat{C}_1$ into (A.12) gives (26). Plugging (26) and $\hat{U}' = -\sigma\hat{C}_1$ into (A.10), we obtain (27). Because (21)–(24) are valid in each $t \in [0, T)$ independent of whether T

is finite or infinite, results (26) and (27) hold in either case. Note that (26) and (27) imply the dynamic relation[15]

$$\hat{c}_1(t) = \psi c_1(t) - \frac{\rho - \psi(1-\sigma)}{\sigma(1+\delta) - \delta}. \quad \textbf{(A.13)}$$

A.6. PROOF OF LEMMA 4

Suppose that $T = \infty$. The dynamic relation (A.13) is globally unstable with a unique fixed point

$$\bar{c}_1 = \frac{1}{\psi} \cdot \frac{\rho - \psi(1-\sigma)}{\sigma(1+\delta) - \delta}. \quad \textbf{(A.14)}$$

Explosive dynamics of $c_1(t)$ can be ruled out, as they lead to either negative consumption or negative output in finite time. The optimal propensity $c_1^*(t)$ is thus equal to $\bar{c}_1$ in each $t \in [T, \infty)$, which proves result (30). From (27) and (30), the depletion rate ϕ coincides with ψc_1^*, so that $\hat{R} = -\psi c_1^* < 0$. Given a constant propensity c_1^*, output grows at the same rate as consumption, and (26) implies (31). Equation (32) follows directly from (4). Restriction (33) and the first inequality in (34) guarantee $c_1^* > 0$ in (30), whereas $c_1^* < 1$ requires the respect of the second inequality in (34). ■

A.7. DERIVATION OF (35)

With slight abuse of notation, rewrite (20) as

$$\mu_1(T) = \mu_2(T). \quad \textbf{(A.15)}$$

Substituting (21) and (23) into (A.15), we obtain (35).

A.8. PROOF OF LEMMA 5

The first step is to derive an explicit expression for the gap function (41). From (21) and (11), we have $\mu_i \psi A d_i = U'(C_i) \cdot e^{-\rho t} Y_i d_i$ in each phase $i = 1, 2$. Substituting these conditions into (18) and (10), the Hamiltonians of the two subproblems evaluated at the switching instant T read

$$H_1(T) = e^{-\rho T}\left[U(C_1(T)) + U'(C_1(T))\, Y_1(T)\, d_1(T)\right] - \lambda(T)\, R(T),$$
$$H_2(T) = e^{-\rho T}\left[U(C_2(T)) + U'(C_2(T))\, Y_2(T)\, d_2(T)\right].$$

Substituting $\lambda(T)R(T) = \delta U'(C_1(T))C_1(T)e^{-\rho T}$ from (22), and using $c_i = 1 - d_i$ to substitute $U'(C_i)Y_i d_i = -U'(C_i)C_i + U'(C_i)Y_i$ into each phase $i = 1, 2$, we have

$$H_1(T) = e^{-\rho T}\left[U(C_1) - U'(C_1)\, C_1 + U'(C_1)\, Y_1 - \delta U'(C_1)\, C_1\right], \quad \textbf{(A.16)}$$
$$H_2(T) = e^{-\rho T}\left[U(C_2) - U'(C_2)\, C_2 + U'(C_2)\, Y_2\right], \quad \textbf{(A.17)}$$

where all variables are evaluated at T. From (35), satisfying condition (20) requires $U'(C_1(T))Y_1(T) = U'(C_2(T))Y_2(T)$, so that the difference between (A.16) and (A.17) equals

$$H_1(T) - H_2(T) = e^{-\rho T}\left[U(C_1) - U'(C_1)\, C_1(1+\delta) - U(C_2) + U'(C_2)\, C_2\right], \quad \textbf{(A.18)}$$

where consumption levels are evaluated at T. Notice that (A.18) is also valid in the limiting case $\sigma = 1$. Now assume that σ differs from unity. Plugging $U(C_i) = (C_i^{1-\sigma} - 1)(1-\sigma)^{-1}$ and $U'(C_i)C_i = C_i^{1-\sigma}$ in (A.18), we can rewrite the gap function (41) as

$$\Omega(T) = e^{-\rho T} \frac{\sigma}{1-\sigma} \left[C_1(T)^{1-\sigma} - C_2(T)^{1-\sigma} - (\delta/\sigma)(1-\sigma) C_1(T)^{1-\sigma} \right]. \qquad \textbf{(A.19)}$$

Next rewrite the terminal condition (35) as

$$\frac{C_1(T)}{C_2(T)} = \left(\frac{Y_1(T)}{Y_2(T)} \right)^{\frac{1}{\sigma}} = \left[\frac{(nR(T))^{\delta}}{\alpha (mG)^{\gamma}} N^{\gamma-\delta} \right]^{\frac{1}{\sigma}}, \qquad \textbf{(A.20)}$$

where the last term is obtained by substituting technologies (1) and (2). From (29) we also know that $R(T) = S_0 \phi (e^{\phi T} - 1)^{-1}$. Substituting this result into (A.20), and collecting the constant terms in $\beta \equiv (nS_0\phi)^{\delta} N^{\gamma-\delta} \alpha^{-1} (mG)^{-\gamma}$, we obtain

$$C_1(T)/C_2(T) = \beta^{1/\sigma} \left(e^{\phi T} - 1 \right)^{-\delta/\sigma}. \qquad \textbf{(A.21)}$$

Plugging (A.21) into (A.19) to eliminate $C_1(T)$, the gap function reads

$$\Omega(T) = \frac{\sigma C_2(T)^{1-\sigma} e^{-\rho T}}{1-\sigma} \left\{ [1 - (\delta/\sigma)(1-\sigma)] \beta^{\left(\frac{1-\sigma}{\sigma}\right)} \left(e^{\phi T} - 1 \right)^{-\frac{\delta(1-\sigma)}{\sigma}} - 1 \right\}. \qquad \textbf{(A.22)}$$

We can now check the existence of an interior switching time: imposing $\Omega(T) = 0$ in (A.22) we obtain

$$e^{\phi T} = 1 + [1 - (\delta/\sigma)(1-\sigma)]^{\frac{\sigma}{\delta(1-\sigma)}} \beta^{1/\delta}. \qquad \textbf{(A.23)}$$

The left-hand side of (A.23) is a strictly increasing function of T, whereas the right-hand side is a positive constant independent of T. Hence, there exists a unique value $T = T'$ satisfying (A.23): taking logarithms of both sides, we obtain

$$T' \equiv \frac{1}{\phi} \ln \left\{ 1 + [1 - (\delta/\sigma)(1-\sigma)]^{\frac{\sigma}{\delta(1-\sigma)}} \beta^{1/\delta} \right\}, \qquad \textbf{(A.24)}$$

where T' is finite and strictly positive because (33) and (34) imply $\phi > 0$ and $[1-(\delta/\sigma)(1-\sigma)] > 0$.[16] Expression (A.24) defines a unique switching instant $T = T'$ associated with the critical condition $\Omega(T') = 0$. We now prove that $T = T'$ is actually the maximum of $V(T)$ by showing that $V(T)$ is strictly increasing in any $T < T'$ and strictly decreasing in any $T > T'$. Rewrite (A.22) as

$$\Omega(T) = \sigma C_2(T)^{1-\sigma} e^{-\rho T} \cdot \left[\frac{f(T) - 1}{1-\sigma} \right], \qquad \textbf{(A.25)}$$

where $f(T) \equiv [1-(\delta/\sigma)(1-\sigma)]\beta^{\left(\frac{1-\sigma}{\sigma}\right)}(e^{\phi T} - 1)^{-\frac{\delta(1-\sigma)}{\sigma}}$. The sign of $\Omega(T)$ is determined by the term in square brackets in (A.25). First suppose that $\sigma < 1$. In this case, $f(T)$ is strictly decreasing in T. Because $f(T') = 1$ by (A.23), we have $f(T'') > 1$ for any $T'' < T'$, and $f(T''') < 1$ for any $T''' > T'$. Given $\sigma < 1$, this implies that $\Omega(T'') > 0$ for any $T'' < T'$, and $\Omega(T''') < 0$ for any $T''' > T'$. Now suppose that $\sigma > 1$ instead. In this case, $f(T)$ is strictly increasing in T, so that $f(T'') < 1$ for any $T'' < T'$, and $f(T''') > 1$ for any $T''' > T'$. Given $\sigma > 1$, this implies again $\Omega(T'') > 0$ for any $T'' < T'$, and $\Omega(T''') < 0$ for any $T''' > T'$. These results imply that $V(T)$ is strictly concave and that

$T = T'$ is the maximum of $V(T)$. We can thus set $T^* = T'$ in (A.25) to obtain (42). The proof of Lemma 5 is completed by considering logarithmic preferences, $\sigma = 1$. Going back to equation (A.18), we can substitute $U(C_i) = \ln C_i$ and $U'(C_i) = C_i^{-1}$ in both phases $i = 1, 2$ to obtain

$$H_1(T) - H_2(T) = e^{-\rho T}\left[\ln\left(\frac{C_1(T)}{C_2(T)}\right) - \delta\right], \quad \sigma = 1. \tag{A.26}$$

The terminal condition (A.21) reduces to

$$C_1(T)/C_2(T) = \beta\left(e^{\phi T} - 1\right)^{-\delta}, \quad \sigma = 1. \tag{A.27}$$

Plugging (A.27) into (A.26), and imposing $H_1(T^*) - H_2(T^*) = 0$, we obtain $e^{\phi T^*} = 1 + (\beta^{1/\delta}/e)$, from which $T^* = (1/\phi)\ln[1 + (\beta^{1/\delta}/e)]$. ■

A.9. PROOF OF LEMMA 6

From the proof of Lemma 5, the optimal switching time T^* is characterized by $\Omega(T^*) = 0$. From (A.19), setting $\Omega(T^*) = 0$ implies that

$$C_1(T^*)/C_2(T^*) = \left\{\sigma\left[\sigma - \delta(1-\sigma)\right]^{-1}\right\}^{\frac{1}{1-\sigma}}. \tag{A.28}$$

Combining (A.28) with condition (A.20), we have

$$Y_1(T^*)/Y_2(T^*) = \left\{\sigma\left[\sigma - \delta(1-\sigma)\right]^{-1}\right\}^{\frac{\sigma}{1-\sigma}}. \tag{A.29}$$

Taking the ratio between (A.28) and (A.29), the optimal ratio between consumption propensities is

$$c_1(T^*)/c_2(T^*) = \sigma\left[\sigma - \delta(1-\sigma)\right]^{-1}. \tag{A.30}$$

From Lemma 1, we can use (14) to substitute $c_2(T^*) = c_2^*$ in (A.30), obtaining

$$c_1(T^*) = \frac{1}{\psi}\cdot\frac{\rho - \psi(1-\sigma)}{\sigma(1+\delta) - \delta}. \tag{A.31}$$

Recalling the derivation of equations (26) and (27), the optimal path of the consumption propensity in the resource-based economy must satisfy the dynamic relation (A.13). Expression (A.31) implies that the optimal consumption propensity at the switching instant, $c_1(T^*)$, must be equal to the steady-state point $\bar{c}_1$ of (A.13)—see equation (A.14). Because (A.13) is globally unstable, the only way to satisfy (A.31) is to set $c_1(t) = \bar{c}_1$ in each instant $t \in [0, T^*)$. As a consequence, the optimal anpath is characterized by a constant propensity to consume, c_1^*, given by (30).[17] From (26), this implies that output and consumption grow at the constant rate (31), and knowledge grows at the constant rate (32) in each $t \in [0, T^*)$. Recalling that (27) and (A.31) imply $\dot{R}/R = -\phi = -\psi c_1^* < 0$, the optimal path is well-defined if and only if parameters satisfy (33) and (34). ■

A.10. PROOF OF PROPOSITION 7

The equations appearing in Proposition 7 are given by (A.28), (A.29), and (A.30), respectively. ■

A.11. DERIVATION OF (43)

By definition, the ratio between the marginal productivities $\partial Y_1/\partial R$ and $\partial Y_2/\partial G$ equals $(\partial Y_1/\partial R)/(\partial Y_2/\partial G) = (\delta Y_1 G)/(\gamma Y_2 R)$. Plugging this expression into (A.29), we have

$$\frac{\partial Y_1 (T^*) / \partial R (T^*)}{\partial Y_2 (T^*) / \partial G} = \frac{\delta G}{\gamma R (T^*)} \left\{\sigma \left[\sigma - \delta (1 - \sigma)\right]^{-1}\right\}^{\frac{\sigma}{1-\sigma}} . \qquad \textbf{(A.32)}$$

From (29) and (A.23), we can respectively substitute $R(T^*) = S_0\phi(e^{\phi T^*} - 1)^{-1}$ and $e^{\phi T^*} - 1 = [1 + (\delta/\sigma)(1 - \sigma)]^{\frac{\sigma}{\delta(1-\sigma)}} \beta^{1/\delta}$, to obtain

$$\frac{\partial Y_1 (T^*) / \partial R (T^*)}{\partial Y_2 (T^*) / \partial G} = \beta^{1/\delta} \frac{\delta G}{\gamma S_0 \phi} \left\{\frac{\sigma - \delta (1 - \sigma)}{\sigma}\right\}^{\frac{\sigma(1-\delta)}{\delta(1-\sigma)}} .$$

Substituting the definition of β from Lemma 5, we obtain expression (43).

A.12. DERIVATION OF (47)

From (11) and (21), we have

$$\mu_2 (T^*) - \mu_1 (T^*) = \frac{e^{-\rho T^*}}{\psi A (T^*)} \left[U' (C_2 (T^*)) Y_2 (T^*) - U' (C_2 (T^*)) Y_1 (T^*)\right].$$

Under logarithmic preferences ($\sigma = 1$), the term in square brackets reduces to $[(1/c_2^*) - (1/c_1^*)]$. Plugging this result in (46) and rearranging terms yields (47).